A guided clothoid spline

D.S. Meek * and R.S.D. Thomas *

Department of Computer Science and Department of Applied Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada

Received January 1990
Revised October 1990

Abstract


A method is described whereby a $G^2$ planar curve consisting of clothoid (Cornu spiral) segments, circular arcs and straight line segments is constructed. The curve passes through given points, is expressed parametrically, and has the attractive feature that the arc length and curvature are piecewise linear functions of the parameter. The given points through which the curve passes must be restricted somewhat so that a unique clothoid exists between each pair of points.

Keywords. Clothoid spline, piecewise linear curvature

1. Introduction

The clothoid is a spiral defined parametrically in terms of Fresnel integrals as

\[
\begin{pmatrix}
    x(t) \\
    y(t)
\end{pmatrix} = \pi B \begin{pmatrix}
    C(t) \\
    S(t)
\end{pmatrix},
\]

(1)

where the scaling factor $\pi B$ is positive, the parameter $t$ is nonnegative, and the Fresnel integrals are [Abramowitz and Stegun '65, p. 300]

\[
C(t) = \int_0^t \cos \frac{1}{2} \pi u^2 \, du, \quad \text{and} \quad S(t) = \int_0^t \sin \frac{1}{2} \pi u^2 \, du.
\]

A recent paper [Heald '85] gives convenient rational approximations to the Fresnel integrals. The clothoid (1) is in the first quadrant, starts at the origin with $t = 0$, and approaches the limiting point \((\frac{1}{2} \pi B, \frac{3}{4} \pi B)\) as $t$ approaches infinity (see Fig. 1). Some useful formulae for the clothoid are that the angle between the tangent vector and the $X$-axis is $\frac{1}{2} \pi t^2$, the curvature is $t/B$, and an element of arc length is $ds = \pi B \, dt$. Below, $t$ is restricted to a range which keeps the angle between the tangent vectors at the endpoints to less than $\pi$ in each clothoid segment.

In this paper, a method is described whereby a $G^2$ planar curve consisting of clothoid (also called Cornu spirals, [Cornu '74]) segments, circular arcs, and straight line segments is constructed. Such a curve will be called a clothoid spline as circles and straight lines can be considered limiting forms of clothoids. The clothoid spline is used in the route design of

* This research was supported by grants from the Natural Sciences and Engineering Research Council of Canada, and the authors acknowledge the hospitality of University of Victoria and Simon Fraser University, respectively, while they were on sabbatical.
highways and railways [Baass '84]. The clothoid segments are used as transitional curves between two different lines, a line and a circular arc, or two circular arcs of different radii. Automatic ways of finding such curves have been described in [Meek & Walton '89] and [Walton & Meek '89], but in those references the spline touches given circles rather than passing through given points. The curve constructed here passes through given points. The curve is expressed parametrically, and has the attractive feature that the arc length and curvature are piecewise linear functions of the parameter. One disadvantage is that the given points through which the curve passes must be restricted somewhat so that a unique clothoid exists between each pair of points.

A basic problem, suggested by [Stoer '82], and referred to as the 'clothoid segment' problem below, is to find a clothoid segment which joins two given distinct points P and Q, and which has given unit tangent vector and signed curvature at P. Once this problem is solved, it is easy to construct a clothoid spline which passes through given points. To find a clothoid spline through P, Q, and R, given the unit tangent vector and signed curvature at P, solve the 'clothoid segment' problem for the points P and Q, calculate the unit tangent vector and signed curvature at Q, and solve the 'clothoid segment' problem for Q and R. This process can be continued, and obviously produces a clothoid spline which is guided by, and passes through, a set of given points. The 'clothoid segment' problem as stated above does not have a unique solution [Stoer '82]. In this paper, it is shown to have a unique solution if Q is restricted to an appropriate 'allowable' region. The 'allowable' region is the region which can be reached by clothoid segments which are restricted so that the tangent vector to the segment rotates through an angle of less than π from one end of the segment to the other. This is not a severe restriction since one would not want the tangent vector of a curve to rotate too much between any two adjacent guiding points.

Conventions to be used below are now mentioned. The direction of increasing arc length is from P to Q. The term tangent will be taken to mean a tangent vector which points in the direction of increasing arc length. The curvature is considered to have a positive sign if the centre of curvature is on the left when moving in a direction of increasing arc length, and a negative sign if the centre of curvature is on the right. Given a finite curve segment of nonnegative curvature whose tangent rotates through an angle of not more than π, extend the
curve with tangential rays from its endpoints so that the extended curve divides the plane into two regions. These two regions, the concave side and the convex side of the extended curve, will be called the concave side and convex side of the curve segment.

A standard form of the 'clothoid segment' problem is obtained by placing $P$ at the origin, aligning the unit tangent vector at $P$ along the positive $X$-axis, and reflecting across the $X$-axis if necessary so that $Q$ is on or above the $X$-axis. The solution of the 'clothoid segment' problem is effected more easily by considering several cases. Let $K$ be the signed curvature at $P$. If $K = 0$, it is clear from Fig. 1 that $Q$ must be in the first quadrant. If $K = 0$ and $Q$ is on the positive $X$-axis, then the straight line segment joining $P$ to $Q$ is the required clothoid segment.

The case of $K = 0$ with $Q$ above the positive $X$-axis is treated in Section 2. If $K < 0$ and $Q$ is on or above the $X$-axis, then it is impossible to join $P$ to $Q$ with a clothoid segment whose tangent rotates less than $\pi$. If $K > 0$ and $Q$ is on the $X$-axis, it is impossible to join $P$ to $Q$ with a clothoid segment whose tangent rotates less than $\pi$. The remaining case of $K > 0$ and $Q$ above the $X$-axis is broken into three subcases. Consider the osculating circle at $P$, which has centre $(0, 1/K)$ and radius $1/K$. The case of $Q$ inside that circle is treated in Section 3, while the case of $Q$ outside that circle is treated in Section 4. If $Q$ is on that circle, the arc of that circle from $P$ to $Q$ is the required clothoid.

2. Curvature at $P$ is zero

In this case, $Q$ is in the first quadrant above the positive $X$-axis. Define the family of clothoid segments, $0 < B < \infty$,

$$D(t, B) = \pi B \begin{pmatrix} C(t) \\ S(t) \end{pmatrix}, \quad 0 < t < \sqrt{2},$$

(2)

where the restriction on $t$ limits the rotation of the tangent to not more than $\pi$. Define the region $R$ as the interior of the wedge-shaped region (see Fig. 2) with apex at the origin $P$, and

![Fig. 2. The 'allowable' region when the curvature at $P$ is zero.](image)
bounded by the ray from the origin, \( D(\sqrt{2}, B), \) \( 0 < B < \infty, \) and the positive \( X \)-axis. The following theorem shows that if a given point \( Q \) is in \( R, \) then a unique \( B \) can be found so that the clothoid segment \( D(t, B) \) passes through \( Q. \)

**Theorem 1.** If \( Q \) is in \( R, \) there is a unique solution to the equation \( D(t, B) = Q. \)

**Proof.** The function \( T(t) = S(t)/C(t), \) with \( T(0) \) defined to be 0, is a strictly monotone increasing function of \( t, 0 \leq t \leq \sqrt{2}. \) Writing \( Q = (Q_1, Q_2)^T, Q \) in \( R \) means that \( T(0) < Q_2/Q_1 < T(\sqrt{2}). \) Thus, there is a unique \( t = t_1, 0 < t_1 < \sqrt{2}, \) for which \( T(t_1) = Q_2/Q_1. \) The scaling factor of the clothoid which passes through \( Q \) is \( aB = Q_2/S(t_1). \)

It is easy to test any given point \( Q \) to see if it is in the ‘allowable’ region \( R. \) Once this is done, the value of \( t \) for which \( T(t) = Q_2/Q_1, 0 < t < \sqrt{2}, \) can be found by using a bisection process (see, for example, [Burden & Faires '85, p. 28]). The corresponding \( B \) can be found as in Theorem 1.

3. Point \( Q \) is inside the osculating circle at \( P \)

In this case, \( K \) is positive and \( Q \) is inside the osculating circle of curvature at \( P. \) Define the family of clothoid segments, \( 0 < B < \infty, \)

\[
E(t, B) = \pi B R(-K^2B^2) \begin{pmatrix} C(t) - C(KB) \\ S(t) - S(KB) \end{pmatrix}, \quad KB < t < \sqrt{K^2B^2 + 2},
\]

where the restriction on \( t \) limits the rotation of the tangent to not more than \( \pi, \) and where \( R(\omega) \) is the matrix of rotation by \( \frac{1}{2}\pi\omega, \)

\[
R(\omega) = \begin{pmatrix} \cos \frac{1}{2}\pi\omega & -\sin \frac{1}{2}\pi\omega \\ \sin \frac{1}{2}\pi\omega & \cos \frac{1}{2}\pi\omega \end{pmatrix}.
\]

Define the region \( R \) to be the interior of the crescent-shaped region (see Fig. 3) bounded by the curve \( E(\sqrt{K^2B^2 + 2}, B), 0 < B < \infty, \) and by the semicircle in the first quadrant \( \text{PXY}. \) The following two theorems show that if a given point \( Q \) is in \( R, \) then a unique \( B \) can be found so that the clothoid segment \( E(t, B) \) passes through \( Q. \)

**Theorem 2.** The family of clothoid segments, \( E(t, B), 0 < B < \infty, KB < t < \sqrt{K^2B^2 + 2}, \) completely covers the region \( R. \)

**Proof.** For the derivation of theoretical results, it is convenient to replace \( t \) by \( T \) in (3), where \( t = \sqrt{K^2B^2 + T}. \) Writing \( C(t), C(KB), S(t) \) and \( S(KB) \) as integrals, and making the above substitution,

\[
E(t, B) = \bar{E}(T, B) = \pi B R(-K^2B^2) \int_{KB}^{\sqrt{K^2B^2 + T}} \begin{pmatrix} \cos \frac{1}{2}\pi u^2 \\ \sin \frac{1}{2}\pi u^2 \end{pmatrix} du, \quad 0 < T \leq 2.
\]

Now, bringing the rotation matrix into the integral, and making the substitution \( U = u^2 - K^2B^2, \)

\[
\bar{E}(T, B) = \frac{1}{2} T \int_{0}^{T} \frac{B}{\sqrt{K^2B^2 + U}} \begin{pmatrix} \cos \frac{1}{2}\pi U \\ \sin \frac{1}{2}\pi U \end{pmatrix} dU, \quad 0 < T \leq 2.
\]

\[\text{(5)}\]
Fig. 3. The ‘allowable’ region when $Q$ is inside the osculating circle at $P$.

It is easy to verify that the clothoid segments $E(T, B)$ can approach arbitrarily closely to the boundary of $R$. As $B$ varies from 0 to $\infty$, the clothoid segments vary from a point at the origin $P$ to the semicircle in the first quadrant $PXY$. The endpoint at $T = 0$ stays at the origin, while the endpoint at $T = 2$ traces the dashed curve $E(\sqrt{K^2B^2 + 2}, B)$ in Fig. 3.

To show that the clothoid segments $E(t, B)$ completely cover $R$, it must be shown that the Jacobian of $E(t, B)$ is nonzero. The Jacobian of $E(t, B)$ is a positive multiple of the Jacobian of $E(T, B)$. The partial derivatives of $E(T, B)$ are fairly easy to work out, and the Jacobian of $E(T, B)$ is

$$\frac{-\pi^2 B}{4\sqrt{K^2B^2 + T}} \int_0^T \frac{u}{(K^2B^2 + u)^{3/2}} \sin \frac{1}{2} \pi (T - u) \, du.$$ 

For $0 < T < 2$, the integrand is positive, and the Jacobian is negative for all $0 < T \leq 2$, $0 < B < \infty$. □

**Theorem 3.** If $Q$ is in $R$, there is a unique solution to the equation $E(t, B) = Q$.

**Proof.** A result in [Goursat '04, p. 321] shows that a planar mapping function which maps the boundary of a domain to the boundary of the image in a one-to-one manner, and has a nonzero Jacobian over the domain is a one-to-one mapping between the domain and the image of the domain. Those conditions are satisfied here; so that for any $Q$ in $R$, there is a unique solution to $E(t, B) = Q$ with $0 < B < \infty$, $KB < t < \sqrt{K^2B^2 + 2}$. □

The above theorem shows that the ‘allowable’ region for $Q$ is the region $R$. To find whether $Q$ is in $R$, it is necessary to determine the side of the curve $E(\sqrt{K^2B^2 + 2}, B)$, $0 < B < \infty$, on which $Q$ lies. The next theorem shows that the above curve has positive curvature.

**Theorem 4.** The curvature of the inner boundary of the crescent-shaped region, $E(\sqrt{K^2B^2 + 2}, B)$ as $B$ varies from 0 to $\infty$, is positive.
Proof. It is more convenient to work with $E(2, B)$ of (5) than $F(\sqrt{k^2R^2+2}, B)$. The curvature of $E(2, B)$ is
\[
\frac{\det(v(B), a(B))}{|v(B)|^3},
\]
where
\[
v(B) = \frac{d}{dB} E(2, B), \quad a(B) = \frac{d^2}{dB^2} E(2, B).
\]
It is only necessary to show that $\det(v(B), a(B))$ is positive. The formulae for $v(B)$ and $a(B)$ are
\[
v(B) = \frac{1}{2} \pi \int_0^2 \frac{u}{(K^2B^2 + u)^{3/2}} \left( \cos \frac{1}{2} \pi u \right) \sin \frac{1}{2} \pi u \, du,
\]
and
\[
a(B) = \frac{-3\pi K^2 B}{2} \int_0^2 \frac{\nu}{(K^2B^2 + v)^{5/2}} \left( \cos \frac{1}{2} \pi v \right) \sin \frac{1}{2} \pi v \, dv.
\]
Now, $\det(v(B), a(B))$ is the difference of products of integrals. The products of integrals can be expressed as double integrals with the result that $\det(v(B), a(B))$ is
\[
\frac{3\pi^2 K^2 B}{4} \int_0^2 \int_0^2 \frac{u}{(K^2B^2 + u)^{3/2}} \frac{\nu}{(K^2B^2 + v)^{5/2}} \sin \frac{1}{2} \pi (u - v) \, du \, dv.
\]
The identity,
\[
\int_0^2 \int_0^2 f(u, v) \, du \, dv = \int_0^2 \int_0^2 (f(u, v) + f(v, u)) \, du \, dv,
\]
applied to the above double integral transforms it into
\[
\int_0^2 \int_0^\nu \frac{uv(v-u)}{((K^2B^2 + u)(K^2B^2 + v))^{3/2}} \sin \frac{1}{2} \pi (v - u) \, du \, dv.
\]
The integrand is nonnegative, and the double integral is positive for all positive $B$. Thus, the curvature of $E(\sqrt{k^2B^2+2}, B)$ is positive for $B$ varying from 0 to $\infty$. \qed

The tangent to $E(\sqrt{k^2B^2+2}, B)$ as $B$ moves away from 0 is a positive multiple of $(C(\sqrt{2}), S(\sqrt{2}))^T$, and the tangent as $B$ approaches $\infty$ is a positive multiple of $(-2, \pi)^T$. The tangent rotates through an angle less than $\pi$, and the curve has positive curvature; so the algorithm WhichSide given in the Appendix can be used to determine the side of $E(\sqrt{k^2B^2+2}, B)$ on which the point $Q$ lies. If $Q$ is on the convex side of that boundary and inside the osculating circle at $P$, then $Q$ is in $R$.

The following idea can be used as the basis for an algorithm to find the scaling factor of the clothoid segment $E(t, B)$ which passes through $Q$. Since each clothoid segment has positive curvature and has a tangent which rotates through an angle of less than $\pi$, the algorithm WhichSide can determine whether $Q$ is on the concave side of, on the convex side of, or on the clothoid segment $E(t, B)$ for any given $B$. If $Q$ is on the concave side of $E(t, B)$, a smaller $B$ is needed, while, if $Q$ is on the convex side of $E(t, B)$, a larger $B$ is needed.

A simple modification of a bisection algorithm can be used to find the scaling factor of the clothoid segment which passes through $Q$. The range of $B$ is 0 to $\infty$, but $E(t, B)$ cannot be
evaluated at either of the extreme values. A finite range for $B$ can be found by the following search. Start with $B = 1/K$ as an initial guess. If a larger $B$ is needed, double $B$, while if a smaller $B$ is needed, halve $B$. This way, the extreme values 0 and $\infty$ can be approached. Eventually, finite positive upper and lower bounds on $B$ will be found, and the standard bisection algorithm can be applied to find an accurate value for $B$.

4. Point $Q$ is outside the osculating circle at $P$

In this case, $K$ is positive and $Q$ is outside the osculating circle at $P$. The clothoid segment will have decreasing curvature as it goes from $P$ to $Q$. Thus, the parameter $t$ will decrease with increasing arc length. This fact is emphasized by the order of the limits on $t$ in (7) and (8). With the angle through which the tangent rotates restricted to less than $\pi$, the region which can be reached by clothoid segments is rather limited. To get a larger region, define an extended clothoid as the clothoid segment which starts at $P$, continues to the point where its curvature is zero, and then is extended by the tangential ray from that point (see Fig. 4). An extended clothoid has a curved part and a straight part.

Define the family of extended clothoids for $0 < KB \leq \sqrt{2}$,

$$
F(t, B) = \begin{cases} 
\pi BR(K^2B^2) \left( \frac{-C(t) + C(KB)}{S(t) - S(KB)} \right), & KB > t > 0 \text{ (curved part)}, \\
\pi BR(K^2B^2) \left( \frac{C(KB) - t}{-S(KB)} \right), & 0 \geq t \text{ (straight part)},
\end{cases}
$$

and the family of clothoid segments for $KB > \sqrt{2}$,

$$
F(t, B) = \pi BR(K^2B^2) \left( \frac{-C(t) + C(KB)}{S(t) - S(KB)} \right), \quad KB > t \geq \sqrt{K^2B^2 - 2},
$$

where the restriction on $t$ in the curved part of extended clothoids and in clothoid segments limits the rotation of the tangent to not more than $\pi$, and where $R(\omega)$ is the matrix of rotation as defined in (4). Define the region $R$ (see Fig. 5) to be the interior of the region bounded by the positive $X$-axis, the semicircle in the first quadrant $PXY$, the curve from $Y$ to $Z$, $F(\sqrt{K^2B^2 - 2}, B), \infty > B > \sqrt{2}/K$, and the ray from $Z$ parallel to the negative $X$-axis. The following two theorems show that if $Q$ is a given point in $R$, then a unique $B$ can be found so that the extended clothoid or clothoid segment $F(t, B)$ passes through $Q$. 
The following theorem shows that if a given point \( Q \) is in \( R \), then a \( B \) can be found so that the extended clothoid or clothoid segment \( F(t, B) \) passes through \( Q \). The proof of this theorem follows the pattern of the proof of Theorem 2.

**Theorem 5.** The extended clothoids \( F(t, B) \) with \( 0 < KB \leq \sqrt{2} \), and the clothoid segments \( F(t, B) \) with \( KB > \sqrt{2} \), completely cover the region \( R \).

**Proof.** For the curved parts of extended clothoids and clothoid segments \( F(t, B) \), it is convenient to replace \( t \) by \( T \), where \( t = \sqrt{K^2B^2 - T} \). Writing \( C(t), C(KB), S(t) \) and \( S(KB) \) as integrals, and making the above substitution,

\[
\tilde{F}(T, B) = \pi B \int_0^{KB} \frac{\cos \frac{1}{2} \pi u^2}{\sqrt{K^2B^2 - T}^2} \sin \frac{1}{2} \pi u^2 \, du, \quad 0 < T \leq \min(K^2B^2, 2).
\]

Now, bringing the rotation matrix into the integral, and making the substitution \( U = K^2B^2 - u^2 \),

\[
\tilde{F}(T, B) = \frac{1}{2} \pi \int_0^T \frac{B}{\sqrt{K^2B^2 - U}} \left( \cos \frac{1}{2} \pi U, \sin \frac{1}{2} \pi U \right) \, dU, \quad 0 < T \leq \min(K^2B^2, 2).
\]

It is easy to verify that \( F(t, B) \) can approach arbitrarily closely to the boundary of \( R \). As \( KB \) varies from 0 to \( \sqrt{2} \), the extended clothoids have one endpoint fixed at \( P \) and vary from the \( X \)-axis to the extended clothoid with \( KB = \sqrt{2} \), which includes the ray from \( Z \) parallel to the negative \( X \)-axis. As \( KB \) varies from \( \sqrt{2} \) to \( \infty \), clothoid segments have one endpoint fixed at \( P \) while the other traces the dashed curve in Fig. 5, \( F(\sqrt{K^2B^2 - 2}, B) \). As \( B \) becomes large, the clothoid segment approaches the semicircle in the first quadrant \( PXY \).

To show that extended clothoids and clothoid segments \( F(t, B) \) completely cover \( R \), it must be shown that the Jacobian of \( F(t, B) \) is nonzero. For curved parts of extended clothoids and
clothoid segments. The Jacobian of $F(t, B)$ is a negative multiple of the Jacobian of $\tilde{F}(T, B)$ in (9). The Jacobian of $\tilde{F}(T, B)$ is

$$\frac{-\pi^2 B}{4\sqrt{K^2 B^2 - T}} \int_0^T \frac{u}{(K^2 B^2 - u)^{3/2}} \sin \frac{1}{2} \pi (u - T) \, du.$$  

For $0 < T < \min(K^2 B^2, 2)$, the integrand is negative, and the Jacobian of $\tilde{F}(T, B)$ is positive.

Theorem 6. If $Q$ is in $R$, there is a unique solution to the equation $F(t, B) = Q$.

Proof. The result of Goursat mentioned in the proof of Theorem 3 can be used again. Here, the boundary is mapped in a one-to-one manner, and the Jacobian is nonzero; so, for any $Q$ in $R$, there is a unique solution to $F(t, B) = Q$. \(\square\)

The first term of this expression is negative or zero as $t \leq 0$, while the second is negative. Thus, the Jacobian of $F(t, B)$ is negative for the straight parts of extended clothoids. \(\square\)

It is seen from Fig. 5 that the only hard part of finding out whether $Q$ is in $R$, is determining the side of the curve $F(\sqrt{K^2 B^2 - 2}, B)$, $\infty > K B > \sqrt{2}$, on which the point $Q$ lies. The next theorem shows that $F(\sqrt{K^2 B^2 - 2}, B)$ is a curve with positive curvature.

Theorem 7. The curvature of part of the boundary of region $R$, $F(\sqrt{K^2 B^2 - 2}, B)$ as $K B$ varies from $\infty$ to $\sqrt{2}$, is positive.

Proof. The proof of this theorem follows the pattern set out in Theorem 4, and is omitted. \(\square\)

The tangent to $F(\sqrt{K^2 B^2 - 2}, B)$ as $K B$ moves from $\infty$ is a positive multiple of $(-2, \pi)^T$, and the tangent as $K B$ approaches $\sqrt{2}$ is a positive multiple of $(-1, 0)^T$. The tangent rotates through an angle less than $\pi$, and the curve has positive curvature; so the algorithm WhichSide given in the Appendix can be used to determine the side of $F(\sqrt{K^2 B^2 - 2}, B)$ on which the point $Q$ lies. Once it has been verified that $Q$ is in $R$, a simple modification of a bisection algorithm, like the one described in Section 3, can be used to find the scaling factor of the clothoid which joins $P$ to $Q$ with the required tangent and curvature at $P$. A starting value for $B$ could be $\frac{1}{2} K$. One small difference between the algorithm here and the one in Section 3 is that if $Q$ is on the concave side of a clothoid segment or extended clothoid, then a larger $B$ is required, while, if $Q$ is on the convex side, a smaller $B$ is required.

5. Example and remarks

A simple example of a clothoid spline is given (see Figs. 6 and 7). Table 1 lists the given points through which it passes, the initial curvature and unit tangent vector, and the unit tangent vectors and the curvatures resulting at the subsequent points. There are seven segments in the curve. The first is a clothoid segment given by equation (2), $0 < t < 0.436$. $B = 36.80$, the
second is an extended clothoid given by equation (7), \(-0.217 < t < 0.364\), \(B = 30.67\), \(K = 0.011860\), the third is a straight line segment, the fourth is a clothoid segment given by equation (2), \(0 < t < 0.336\), \(B = 62.11\), the fifth is a circular arc, the sixth is a mirror image across the \(X\)-axis of a clothoid segment given by equation (3), \(0.194 < t < 0.702\), \(B = 35.83\), \(K = 0.005414\), and the seventh is a circular arc.

Some remarks on drawing clothoid splines are in order. Firstly, the method described here is not a general curve drawing method because the guiding points cannot be chosen arbitrarily. The sensible way to use the method is to use it interactively choosing the guiding points one at

Table 1
Data and results for the example clothoid spline

<table>
<thead>
<tr>
<th>point</th>
<th>unit tangent vector</th>
<th>curvature (K)</th>
<th>joining curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.00, 0.00)</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.</td>
</tr>
<tr>
<td>(50.00, 5.00)</td>
<td>0.9556</td>
<td>0.2947</td>
<td>0.011860 clothoid segment, Section 2</td>
</tr>
<tr>
<td>(100.00, 30.00)</td>
<td>0.8742</td>
<td>0.4855</td>
<td>0.</td>
</tr>
<tr>
<td>(120.96, 41.64)</td>
<td>0.8742</td>
<td>0.4855</td>
<td>0.</td>
</tr>
<tr>
<td>(180.03, 70.01)</td>
<td>0.9462</td>
<td>0.3234</td>
<td>-0.005414 clothoid segment, Section 2</td>
</tr>
<tr>
<td>(236.02, 79.90)</td>
<td>0.9997</td>
<td>0.0203</td>
<td>-0.005414 circular arc</td>
</tr>
<tr>
<td>(290.00, 65.00)</td>
<td>0.7677</td>
<td>-0.6406</td>
<td>-0.019597 clothoid segment, Section 3</td>
</tr>
<tr>
<td>(305.21, 43.41)</td>
<td>0.3447</td>
<td>-0.9386</td>
<td>-0.019597 circular arc</td>
</tr>
</tbody>
</table>
Fig. 8. The initial test.

a time. That way, the guiding points can be chosen in the 'allowable' region at each step. The method is quite useful in designing curves which have piecewise linear curvature, as highway and railway routes do.

Appendix

The following algorithm is an efficient way to determine whether a given point is on the concave side of, on the convex side of, or on a curve segment which is given parametrically, whose curvature is nonnegative, and whose tangent rotates through an angle of not more than π.

Algorithm WhichSide

Assume that the curve segment goes from the point $S$ to the point $E$, and the point to test is $Q$. If $Q$ is on the tangent line through $S$ or the tangent line through $E$, it is easy to decide whether $Q$ is on the extended curve or on the convex side of the curve. Now, assuming $Q$ is not on either of those tangent lines, make the following initial check (see Fig. 8). If $Q$ is to the left of the tangent at $S$, to the left of the tangent at $E$, and on or to the left of the line segment $SE$, then $Q$ is on the concave side of the curve. If $Q$ is to the right of the tangent at $S$, or to the right of the tangent at $E$, then $Q$ is on the convex side of the curve. For other positions of $Q$, the 'don't know' triangle, it is not yet known whether $Q$ is on the curve or on the concave or convex side of the curve.

Fig. 9. One step of the iteration.
Now consider the following reduction of a 'don’t know' triangle (see Fig. 9). Suppose A, B, and C are three points on the curve, with B being about midway between A and C. Suppose that Q is inside the triangle bounded by line AC, the tangent line at A, and the tangent line at C. If Q and B are the same point, then Q is on the curve. Otherwise, assuming that Q is not the same point as B, draw the line segment AB, and the line segment BC. If Q is on or to the right of the tangent at B, then Q is on the convex side of the curve. If Q is on AB or on BC, then Q is on the concave side of the curve, and if Q is to the left of AB and to the left of BC, then Q is on the concave side of the curve. Suppose it still has not been decided whether Q is on the concave or convex side of the curve. If Q is to the right of AB, then Q is in the first ‘don’t know’ triangle. If Q is to the right of BC, then Q is in the second ‘don’t know’ triangle. Continue the reduction of the appropriate ‘don’t know’ triangle until a decision can be made, or the reduced ‘don’t know’ triangle containing Q is flat enough that Q can be considered to be on the curve. If it is determined that the point is on the curve or close enough to be considered on the curve, an approximation to the corresponding parameter value is the average of the parameter values at the two vertices of the ‘don’t know’ triangle containing Q which are on the curve.

Acknowledgement

The authors appreciate the referees’ many suggestions which helped improve this paper considerably.

References